CAMBRIDGE

Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2024

Examiners' Report

Mark Scheme

STEP MATHEMATICS 3

2024

Examiners' Report

STEP 3 Introduction

The total entry was an increase on that of 2023 by more than 10%. One question was attempted by more than 98% of candidates, another two by about 80%, and another five by between 50% and 70%.

The remaining four questions were attempted by between 5% and 30% of candidates, these being from Section B: Mechanics, and Section C: Probability and Statistics, though the Statistics questions were in general attempted more often and more successfully.

All questions were perfectly solved by some candidates.
About 84% of candidates attempted no more than 7 questions.

Question 1

This was comfortably both the most popular question and the most successful, with a mean score of about 15/20. There were numerous correct methods employed to approach the partial fractions. Every part had many excellent clear responses. Generally, if candidates could do the partial fractions algorithm correctly and wrote more than the bare minimum for the limiting and telescoping operations they got almost full marks.

In part (i), most could do the calculations correctly, though explanations less so.

In parts (ii) and (iii), many candidates did not attempt the correct decomposition. Explanations of cancelling terms in the telescoping series and taking limits were frequently not clear. Particular weaknesses were treating harmonic series as if they converged, and substituting ∞ into expressions as if it were a number.

There were many clever ways of doing the last part without a full partial fraction decomposition, but probably the cleanest was as follows.

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)^2}-\sum_{r=1}^{\infty}\frac{2}{r^2(r+1)}=\sum_{r=1}^{\infty}\frac{-2r-1}{r^2(r+1)^2}=\sum_{r=1}^{\infty}\frac{r^2-(r+1)^2}{r^2(r+1)^2}$$
$$=\sum_{r=1}^{\infty}\left(\frac{1}{(r+1)^2}-\frac{1}{r^2}\right)=-1$$

Question 2

Three quarters of the candidates attempted this question with a mean score of just under half marks.

In part (i), candidates often omitted a justification that the LHS of the inequality was real and for noting that both sides are positive before squaring.

Part (ii)(a) was generally done quite well, although some candidates ignored the suggested method and argued that because the lead terms cancel as $x \to \infty$, $\mathrm{f}(x) \to 0$, not earning full marks.

The sketch in (ii)(b) was not generally done very well. In general, sketches just need to have the same key features as the actual plot of the function. The asymptotes and symmetry about $x = 1$ were crucial here.

Part (iii) was done fairly well by those that attempted it, most noticing that they should choose values of m to ensure that the x^2 terms should cancel.

There were not many significant attempts on part (iv). To start, it was relatively straightforward to state that as four critical values were required, the quadratic needed to cross the x-axis, but this was often missed. However, there were some very efficient and neat solutions to this part, and candidates who got on the right path initially executed it well. The most common error was failure to get the four roots attached to the correctly signed version of the quadratic. Candidates who used a diagram were generally much more successful with this.

Question 3

The second most popular question, it was the eighth most successful with a mean score of a little under 9/20.

Whilst some candidates did not make progress with differentiating f in (iii), most differentiated well in (i) and (iii).

However in (i), sufficient justification for the positive gradient for $c \geq \frac{1}{2}$ was often missing in (a), and some occasionally forgot that inequalities reverse when divided by a negative number in (b).

In part (ii), both sketch graphs were mostly drawn correctly. However, in part (a), many did not justify the positive gradient or asymptote for large x. In part (b), whilst most found the turning point correctly, few justified the positive gradient before the turning point.

The justifications, or otherwise, in (iii) varied a lot in the level of detail. Forgetting to mention that $\mathrm{f} > 0$ was a common way that candidates did not achieve full marks.

Question 4

The fourth most popular question, it was the third most successful, with a mean score of 10 marks.

Part (i) needed more thoroughness than many attempts displayed. Most sensibly chose to express the gradients as tangents of angles of the lines to the *x*-axis, but then did not define these or consider the possible cases that could arise such as which was greater, or state that the difference between the angles is ±45° or 45°/135°. As the result was given in the question, there was an expectation that there should be complete justification.

In part (ii), most attempts at the coordinates of the point of intersection were successful, though many did not use the non-equality of p and q, and a large number got the y coordinate wrong through substituting x into the equation of the parabola. Overall, many did well with the final result of this part, employing the various results from earlier in the part and that of (i).

Part (iii) proved challenging for most, and there was a fair amount of guesswork based on the knowledge that 30°, 45° and 60° are angles with nice trigonometric values!

Question 5

This question was a little less popular than question 4 but was less successful with a mean score of under 8/20.

The first part was very well-answered with some efficiently realising that elements not on the leading diagonal did not need calculating. Sadly, some overlooked the second result required.

Part (ii) was well-answered too, with the same efficiency as in (i) being employed by some.

Part (iii) was less well-answered, with the non-conjugate nature of matrix multiplication often being overlooked, and in the last result treating A, B, C, and D as constants. Applying the scalar version of the chain rule to differentiate M^2 was not an uncommon error, but those that answered this part successfully usually rewrote tr(M^2) in terms of tr(M) and det(M).

Part (iv) caused the most difficulty. Only a handful attempted to provide an explicit counterexample to the statement. Some gave a counterexample that did not satisfy all the conditions on M and N, and a larger number of students convinced themselves that there is no good reason for the claim to hold, but did not give a counterexample. Some students attempted to prove the claim was true. Due to this there were many more 17/20 solutions than 18 or 19/20 solutions. Only 6 candidates achieved 20/20.

Question 6

This was the least popular of the Pure Mathematics section, and by a large margin the least successful of the whole paper.

Those candidates who were successful in part (i)(a) usually tackled the question by re-writing the differential equation as $\frac{\mathrm{d}(x-y)}{\mathrm{d}t} = -2(x - y)$. There were also some candidates who rewrote the equation as $\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = \frac{\mathrm{d}y}{\mathrm{d}t} = 2y$ and used integrating factors effectively to solve this, although some integrated erroneously to achieve $x + 2xt = y + 2yt$. Some candidates correctly concluded that $x = y$ but did not go on to say that this implied that $x_0 = y_0$. Most of the candidates gaining no credit for this question substituted $x = y = 0$ into their differential equation and then integrated that.

In part (i)(b) those candidates who attempted it generally understood what was required, but some did not appreciate that the situation in this case had different initial conditions to that in part (a). Some candidates used the given differential equations to find a second order differential equation in x or y, which was a valid if inefficient method.

Those attempting part (ii) generally performed in a similar way to part (i), either gaining most of the credit available or making the same mistakes they had made in the previous parts. There were some candidates who rather cleverly spotted that they could combine the last two differential equations to show that $y = z$, and then show that $x = z$ and in so doing answer both parts (ii)(a) and (b) together.

Question 7

The third most popular question, this was a little less successfully attempted than question 2 with a mean score of just over 9/20.

Parts (i) and (ii) were not generally well done, as it was easy to guess the geometric series and then make unsubstantiated, or at least unjustified, claims which could not be given full marks.

In part (ii), there was frequently lack of clarity regarding pairing of terms and arguments lacking in necessary detail to support the claims.

Part (iii) was done better, though the second result commonly saw $1/e$ expanded as a reciprocal rather than as e^{-1}, and then, as a consequence, getting lost.

Part (iv), too, was fairly well done. There was a good understanding of contradiction arguments for part (v), though there was difficulty in choosing a suitable n in quite a few cases.

Question 8

One of the least popular questions in the Pure Mathematics section, candidates did slightly less well here than on question 7. There were some excellent answers to this question, but also some answers that were lacking in clear explanation. There were sometimes issues with candidates not understanding the direction of implication required by the various question parts. The best solutions used the structure of the question to help find appropriate and efficient methods to solve the problem but there were also some inventive solutions using other techniques.

Part (i) was generally done well, though some candidates did not show sufficient working to justify the given answer fully.

Part (ii) was also generally done well, but some candidates did not take advantage of the work done in the previous part to show that the given equation represented a pair of straight lines. A small minority of candidates instead tried to show that if the equation represented a pair of straight lines then $k = 1$.

Parts (iii) and (iv) were found to be more difficult.

In Part (iii) the most successful candidates tended to follow the lead of the previous parts and factorised the equation in part (ii) to find the equations of two straight lines. A considerable number of candidates made a sign error while doing this: expanding to check a factorisation is correct is always a good idea. Those that factorised usually could see how to set up two quadratic equations in x and so find a condition of s. Some candidates set up a quartic equation in x but only a small number of these could complete an argument to show that $s < -0.75$, and these candidates often were confused on the direction of implication needed in this part.

The direction of implication required in part (iv) confused a lot of candidates, with some stating that they had already answered this in the previous part and others repeating a proof that four distinct points implies $s < -0.75$. Some other candidates recognised that there must be two distinct points of intersection of the curves and each line but did not realise that one of these points of intersection could be where both curves and both lines meet. A sketch was often a good idea to help clarify the geometry of the situation. A handful of candidates managed to consider the “if and only if” situation by considering where the two straight lines were tangential to $y = x^2$ answering both of the last two parts in one go.

Question 9

This was an unpopular question, only being attempted by about a seventh of the candidates. It was also the second least successful with a mean score of only 4/20. There were mixed responses, and it mostly depended on how the diagram was set up, that is in which directions candidates chose to label the velocities. Many candidates struggled to understand how to apply the restitution law when the particles collide obliquely rather than directly along the line of centres. Some tried to use total speeds of the particles rather than the speeds along the line of contact, and some tried to use the horizontal speeds. Many also did not use vectors correctly, drawing vectors in certain directions then not introducing necessary negative signs.

Other than that, part (i) was done well and most understood how to rotate the solution back into usual x-y directions.

Those who got to part (ii) generally did it easily.

Most found part (iii) trickier, and it tended to be either done well or not really started. Once the diagram was set up, it was found to be straightforward, and most who got that far saw how to proceed.

There were very few significant attempts at part (iv).

Question 10

This was the least popular question on the paper by some way, being attempted by fewer than 6% of the candidates. It was attempted only a little more successfully than question 9 scoring a mean of about 5.5/20. Some of the few attempts were little more than a poor diagram and nothing further. If it was setup correctly, the candidates did fairly well, despite losing marks for not drawing everything required on the diagram, though there was some leniency about drawing equal and opposite forces (e.g. the reaction force from the top cube down onto the bottom cube). It should be stressed that very few did this so it could be a point of focus when preparing candidates for STEP mechanics. The only common error found once the first part was complete was mostly to do with reading carefully.

In part (iii), most did not check that the upper cube could not topple without the lower toppling first, they just compared toppling of bottom cube and slipping.

The main challenge in this sort of question is in the initial setup, after which the techniques required are not particularly difficult. Candidates who were able to interpret the context and setup the situation usually did very well.

Question 11

Very nearly 30% of the candidates attempted this, making it the most popular non-Pure question, and they did so relatively successfully with a mean score of nearly 11/20, better than all but question 1. A significant number of candidates gained full or close to full credit.

Part (i) was generally well executed, although using $r\binom{2n}{r} = (2n+1-r)\binom{2n}{2n+1-r}$ for $r = 0$ without justification was a common error.

In part (ii), a common error was using an incorrect probability distribution for the random variable X, common examples included asserting that X itself was binomially distributed as $\mathrm{B}\left(2n, \frac{1}{2}\right)$, or asserting that either $\mathrm{P}(X = k) = \frac{1}{2^{2n}}\binom{2n}{k}$ or $\mathrm{P}(X = k) = \frac{2}{2^{2n}}\binom{2n}{k}$ for all $n \leq k \leq 2n$.

Showing that $\frac{1}{2^{2n}}\binom{2n}{n}$ is a decreasing function of n in part (iii) was generally well executed; a few students considered the difference between $\frac{1}{2^{2n}}\binom{2n}{n}$ and $\frac{1}{2^{2n+2}}\binom{2n+2}{n+1}$, rather than the ratio, which lead to a largely similar, but slightly more involved, computation.

Part (iv) commonly saw candidates trying to maximise total expected winnings, rather than expected winnings per pound. However generally the standard of responses to this question was quite high.

Question 12.

A little over one fifth of the candidates attempted this, marginally less successfully than question 11 with a mean score of 10 marks. As with question 11, a significant number of candidates gained full or close to full credit. In the main, there was a dichotomy in student responses: for each of the parts, students were generally either unable to make any real progress with that part question or were able to produce a relatively full solution.

Parts (i) and (ii) were generally well done, although quite a common error was to incorrectly differentiate the cumulative distribution function from (i) to find the probability distribution required for (ii). Another quite common error was attempting to use integration by change of variable rather than by parts to evaluate $\int r^2 \, cos^{-2}(r^{-1}) \, dr$ in (ii).

A number of students only attempted part (iii) of the question, in many of these cases, gaining full or close to full marks. For this part, by far the most common approach was to use the substitution $r = cosh \, u$ to evaluate the integral. However, other solutions were also seen. Various different substitutions were used either successfully, or at least in some way productively, to evaluate the integral, including $r = sec \, u$, $r =$ cosec u , $r = coth \, u$, the double substitution $u = \sqrt{r^2 - 1}$ followed by $u = sinh \, x$, and the double substitution $r = sec \, u$ followed by $x = sin \, u$. However, it was an uncommon to see an unproductive substitution such as $u = r^2 - 1$.

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